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Spaces of Whitney Functions on Cantor-Type Sets

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Abstract. We introduce the concept of logarithmic dimension of a compact set. In terms of this magnitude, the extension property and the diametral dimension of spaces $\mathcal{E}(K)$ can be described for Cantor-type compact sets.

1 Introduction

We introduce a concept of logarithmic dimension as the following generalization of the Hausdorff dimension: take the function $\psi(r) = \frac{1}{\log \frac{1}{r}}$, $r > 0$, corresponding to the logarithmic measure; then for any compact set $K \subset \mathbb{R}$ there exists a critical value $\lambda_0 = \lambda_0(K) \in [0, \infty]$ (called the logarithmic dimension of K) such that for $\lambda < \lambda_0$ the ψ^λ-measure of K is ∞, for $\lambda > \lambda_0$ it is zero.

The aim of this article is to show that for the class $\mathcal{E}(K)$ of Whitney functions defined on generalized Cantor sets the logarithmic dimension is highly suitable for the investigation of the following two problems.

The problem of geometric characterization of the extension property of K (that is the existence of a continuous linear extension operator $L\colon \mathcal{E}(K) \to C^\infty(\mathbb{R}^n)$) was posed by Mityagin [11, Section 8.5]. Some particular results were given by Stein [16], Bierstone [2], Pawłucki and Pleśniak [13] and others. In [17] Tidten suggested to consider perfect sets of class (α) (see [18], [6] for definition) and proved that the condition $K \in (1)$ is sufficient for the extension property of K whereas "$K \in (\alpha)$ for some α" is necessary. In particular he showed that the classical Cantor set C (clearly $\lambda_0(C) = \infty$) has the extension property. Here we show that the generalized Cantor set K has the extension property if $\lambda_0(K) > 1$ and it has not if $\lambda_0(K) < 1$. Examples of perfect sets of finite class without extension property can be given easily (compare this with [6]).

On the other hand, the diametral dimension of the space $\mathcal{E}(K)$ in our case can be described in terms of the logarithmic dimension of K (Chapter 4) and what is more, we have complete (up to the case of equal logarithmic dimensions) isomorphic classification of given spaces $\mathcal{E}(K)$ (Theorem 3). Examples of continua of pairwise nonisomorphic spaces of considered type appear directly (compare with [8]). It should be noted that the diametral dimension can not be applied to distinguish nuclear spaces of type $C^\infty(\overline{\Omega})$ or $\mathcal{E}(K)$ with $\mathring{K} \neq \varnothing$. In fact, all these spaces contain a subspace which is isomorphic to the space s of rapidly decreasing sequences and thus their diametral dimension $\Gamma(X)$ is not larger than $\Gamma(s)$ (see for example [11, Proposition 7]).

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On the other hand, the space s has the smallest diametral dimension in the class of nuclear spaces and we get $\Gamma(X) = \Gamma(s)$.

2 Logarithmic Dimension

We consider the following generalization of the Cantor ternary set. Let $(l_n)_1^\infty$ be a sequence of positive numbers and $(N_n)_1^\infty$ be a sequence of integers, $N_n \geq 2$ for all n. Then $K = K\big((l_n),(N_n)\big) = \bigcap_{n=0}^\infty E_n$, where $E_0 = I_{0,1} = [0,1]$ and E_n, $n \geq 1$, is a union of $N_1N_2\cdots N_n$ disjoint closed intervals $I_{n,k}$ of length l_n and E_{n+1} is obtained by replacing each interval by N_{n+1} disjoint subintervals $I_{n+1,j}$ of length l_{n+1} with $N_{n+1}-1$ gaps of length h_{n+1}. The intervals $I_{n,k}$ that make up the set E_n are called *basic intervals*. The set K is well-defined if for all n we have $l_{n-1} > N_nl_n$ with $l_0 = 1$. Then $h_n = \frac{l_{n-1}-N_nl_n}{N_n-1}$. We will restrict ourselves to the case $l_n \leq h_n$, since otherwise K is uniformly perfect and has the extension property.

Let $\alpha_1 = 1$ and for $n \geq 2$ let α_n satisfy $l_n = l_{n-1}^{\alpha_n}$. For $n \geq 2$ set $\lambda_n = \frac{\log N_n}{\log \alpha_n}$. We will analyze two regular cases: $N_n = N$ for all n and $N_n \nearrow \infty$ as $n \to \infty$. The corresponding compact set of finite type will be denoted by K_N; K_∞ stands for the infinite case. If in particular $\alpha_n = \alpha$, $n \geq 2$, then we will write $K_N^{(\alpha)}$ and $K_\infty^{(\alpha)}$.

Let us introduce a parameter which can be applied in classifying such rarefied sets as $K_N^{(\alpha)}$. Let $\psi(r) = \frac{1}{\log\frac{1}{r}}$, $r > 0$. Here and subsequently, log denotes the natural logarithm, $[x]$ denotes the greatest integer in x. For $0 < \lambda < \infty$ consider the ψ^λ-measure of a compact set K (see *e.g.* [12, V.6.2] or [3], [15]): let $m_\epsilon(K,\psi^\lambda) = \inf \sum \psi^\lambda(r_i)$ where the greatest lower bound is taken over all covers $\bigcup B_i$ of K by balls B_i with diam $B_i = r_i \leq \epsilon$; then $m(K,\psi^\lambda) = \lim_{\epsilon\to 0} m_\epsilon(K,\psi^\lambda)$. As in the definition of the Hausdorff dimension (see *e.g.* [15, 10.1], [3, 1.2]) we see that there exists a critical value $\lambda_0 = \lambda_0(K)$, $0 \leq \lambda_0 \leq \infty$, such that $m(K,\psi^\lambda) = \infty$ for $\lambda < \lambda_0$ and $m(K,\psi^\lambda) = 0$ for $\lambda > \lambda_0$. Since the function ψ corresponds to the logarithmic measure we will say that the value λ_0 is the *logarithmic dimension* of K.

Proposition 1 *Suppose that for* $K = K\big((l_n),(N_n)\big)$ *the limit* $\lambda_0 = \lim_n \lambda_n$ *exists in the set of extended real numbers. Then* λ_0 *is the logarithmic dimension of* K. *In particular,* $\lambda_0(K_N^{(\alpha)}) = \frac{\log N}{\log \alpha}$.

Proof For $\lambda > \lambda_0$ consider $m_\epsilon(K,\psi^\lambda)$. Notice that $\alpha_n \nrightarrow 1$. In fact, otherwise we would have $\lambda_0 = \infty$. For $\epsilon = l_n$ we have the covering of K by $N_1N_2\cdots N_n$ intervals $I_{n,k}$ of length ϵ. Thus

$$m_\epsilon(K,\psi^\lambda) \leq N_1N_2\cdots N_n \log^{-\lambda}\left(\frac{1}{l_n}\right).$$

Since $l_n = l_1^{\alpha_1\alpha_2\cdots\alpha_n}$ and $N_n = \alpha_n^{\lambda_n}$ we get

$$m_\epsilon(K,\psi^\lambda) \leq \alpha_1^{\lambda_1-\lambda}\cdots\alpha_n^{\lambda_n-\lambda} \log^{-\lambda}\left(\frac{1}{l_1}\right),$$

which tends to 0 as $n \to \infty$ since $\lambda > \lim \lambda_n$ and $\alpha_n \nrightarrow 1$. Thus, $m(K,\psi^\lambda) = 0$.

Suppose now that $\lambda < \lambda_0$. Then we can modify the arguments in [3, 2.3]. For $\epsilon > 0$ there exists a finite covering $\bigcup_{i=1}^M U_i$ of K by open intervals U_i, diam $U_i = r_i < 2\epsilon$ such that $\sum \psi^\lambda(r_i) \le 2m_\epsilon(K, \psi^\lambda)$. For each r_i fix $n = n(i) \in \mathbb{N}$ with $l_n \le r_i < l_{n-1}$. Let $n_0 = \min_{i \le M} n(i)$, $n_1 = \max_{i \le M} n(i)$. To simplify calculations we set $l_1 = 1/e$. Then

$$\psi^\lambda(r_i) \ge \psi^\lambda(l_n) \ge (\alpha_1 \cdots \alpha_n)^{-\lambda}.$$

Let ϵ be so small that $\lambda_n > \lambda$ for $n \ge n_0$. Then

$$\begin{aligned}(\alpha_1 \cdots \alpha_n)^\lambda &= (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \cdot (\alpha_{n_0}^\lambda \cdots \alpha_n^\lambda) \\ &\le (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \cdot (\alpha_{n_0}^{\lambda_{n_0}} \cdots \alpha_n^{\lambda_n}) \\ &= (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \cdot N_{n_0} \cdots N_n.\end{aligned}$$

We decompose the sum $\sum \psi^\lambda(r_i)$ into two parts. Let $\sum'$ be the sum over all i such that $l_n \le r_i < \frac{l_{n-1}}{N_n}$, and $\sum''$ be the sum over the remaining i's. Since $\frac{l_{n-1}}{N_n} < l_n + h_n$, for any i in the sum $\sum'$, the interval U_i can intersect at most two basic intervals of E_n. By construction, it can intersect at most $2N_{n+1}$ basic intervals of $E_{n+1}; \cdots ; 2N_{n+1} \cdots N_{n_1}$ basic intervals of E_{n_1}. Then

$$\begin{aligned}2N_{n+1} \cdots N_{n_1} &\le 2N_{n+1} \cdots N_{n_1} \cdot (\alpha_1 \cdots \alpha_n)^\lambda \cdot \psi^\lambda(r_i) \\ &\le 2N_{n_0} \cdots N_{n_1} \cdot (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \cdot \psi^\lambda(r_i).\end{aligned}$$

For i in the second sum $\sum''$, fix j, $j = 1, 2, \ldots, N_n - 1$, such that $\frac{j}{N_n} l_{n-1} \le r_i < \frac{j+1}{N_n} l_{n-1}$. Then the interval U_i can intersect at most $j+2$ basic intervals of E_n and thus $(j+2)N_{n+1} \cdots N_{n_1}$ basic intervals of E_{n_1}. Here

$$\psi^\lambda(r_i) \ge \psi^\lambda\left(\frac{j}{N_n} l_{n-1}\right) \ge \left(\alpha_1 \cdots \alpha_{n-1} + \log \frac{N_n}{j}\right)^{-\lambda}.$$

If $\log \frac{N_n}{j} \ge \alpha_1 \cdots \alpha_{n-1}$, then $1 \le 2^\lambda \log^\lambda(\frac{N_n}{j}) \psi^\lambda(r_i) \le C'_\lambda \frac{N_n}{j} \psi^\lambda(r_i)$. Therefore

$$\begin{aligned}(j+2)N_{n+1} \cdots N_{n_1} &\le C''_\lambda N_n N_{n+1} \cdots N_{n_1} \psi^\lambda(r_i) \\ &\le C''_\lambda N_{n_0} \cdots N_{n_1} \cdot (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \psi^\lambda(r_i).\end{aligned}$$

On the other hand, if $\log \frac{N_n}{j} < \alpha_1 \cdots \alpha_{n-1}$, then $1 \le 2^\lambda (\alpha_1 \cdots \alpha_{n-1})^\lambda \psi^\lambda(r_i)$, therefore

$$\begin{aligned}(j+2)N_{n+1} \cdots N_{n_1} &\le (N_n + 1)N_{n+1} \cdots N_{n_1} (\alpha_1 \cdots \alpha_{n-1})^\lambda 2^\lambda \psi^\lambda(r_i) \\ &\le 2^{\lambda+1} N_{n_0} \cdots N_{n_1} (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \psi^\lambda(r_i).\end{aligned}$$

Thus any interval U_i, $i \le M$ can intersect at most

$$C_\lambda N_{n_0} \cdots N_{n_1} (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \psi^\lambda(r_i)$$

basic intervals of E_{n_1}. Here $C_\lambda = \max\{C''_\lambda, 2^{\lambda+1}\}$. Since the covering $\bigcup U_i$ intersects all basic intervals of E_{n_1}, we have

$$N_1 \cdots N_{n_1} \leq C_\lambda N_{n_0} \cdots N_{n_1} (\alpha_1 \cdots \alpha_{n_0-1})^\lambda \sum \psi^\lambda(r_i)$$

and so

$$\sum \psi^\lambda(r_i) \geq C_\lambda^{-1} N_1 \alpha_2^{\lambda_2-\lambda} \cdots \alpha_{n_0-1}^{\lambda_{n_0-1}-\lambda}.$$

This bound implies that the sum of type $\sum \psi^\lambda(r_i)$ must be arbitrarily large for small enough ϵ, that is $m(K, \psi^\lambda) = \infty$.

In fact, for $\alpha_n \nrightarrow 1$ it is obvious as $\lambda_n > \lambda$ for large n. Clearly, $\epsilon \to 0$ gives $n_0 \to \infty$. If $\alpha_n \to 1$, then $\alpha_n^{\lambda_n - \lambda} = \frac{N_n}{\alpha_n^\lambda} \geq \frac{2}{\alpha_n^\lambda} \to 2$ as $n \to \infty$ and the result follows. ■

The value $\lambda_0 = 1$ is critical in Potential Theory: if $\lambda_0(K) < 1$, then the logarithmic measure of K is 0 and the set K is exceptional. We show that this bound is crucial as well for the extension property of Cantor-type sets.

3 Extension Property

Let $\mathcal{E}(K)$ denote the space of Whitney functions on a perfect compact set K with the topology defined by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(j)}(x)| \cdot |x-y|^{j-q} : x, y \in K, x \neq y, j \leq q\}, \quad q = 0, 1, \ldots,$$

where $|f|_q = \sup\{|f^{(j)}(x)| : x \in K, j \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. Each function $f \in \mathcal{E}(K)$ is extendable to a C^∞-function on the line. If there exists a linear continuous extension operator $L\colon \mathcal{E}(K) \to C^\infty(\mathbb{R})$, then we say that the compact set K has the *extension property*. In [17] Tidten showed that the extension property of K and the property DN of the space $\mathcal{E}(K)$ are equivalent.

A Fréchet space X with a fundamental system of seminorms $(\|\cdot\|_q)$ is said to have the property DN [21] (see also the class D_1 in [23]) if

$$\exists\, p\, \forall q\, \exists\, r, C > 0 : \|\cdot\|_q \leq t\|\cdot\|_p + \frac{C}{t}\|\cdot\|_r, \quad t > 0.$$

Here $p, q, r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Proposition 2 *For $X = \mathcal{E}(K)$, the following statements are equivalent to* DN*:*

(a) $\exists\, p\, \exists\, R > 0\, \forall q\, \exists\, r, C : |\cdot|_q \leq t^{R\cdot q}\|\cdot\|_p + \frac{C}{t^q}\|\cdot\|_r,\ t > 0;$
(b) $\exists\, p\, \forall \epsilon > 0\, \forall q\, \exists\, r, C : |\cdot|_q^{1+\epsilon} \leq C\|\cdot\|_p\, \|\cdot\|_r^\epsilon.$

Proof In [10, Lemma 29.10] and in [5] it was shown that DN is equivalent to the conditions (a), (b) with $\|\cdot\|_q$ instead of $|\cdot|_q$. Frerick [4] and Tidten [19] proved that we can replace $\|\cdot\|_q$ by $|\cdot|_q$.

We first generalize Theorems 2 and 3 in [6].

Theorem 1 *If $\underline{\lim}\,\alpha_n > N$, then K_N does not have the extension property. If $\overline{\lim}\,\alpha_n < N$, then K_N has the extension property.*

Proof Let β be such that $\underline{\lim}\alpha_n > \beta > N$ and find i_0 such that $\alpha_i > \beta$, $i \geq i_0$. Fix $0 < \epsilon < \frac{\beta - N}{2(N-1)}$ and $M \in \mathbb{N}$ such that $M \geq \frac{2\beta}{\beta - N}$. We want to show

$$\forall p\, \exists \epsilon\, \exists q : \forall r > q\, \exists (f_n) \in \mathcal{E}(K_N) \colon \frac{\|f_n\|_p \cdot \|f_n\|_r^{\epsilon}}{|f_n|_q^{1+\epsilon}} \to 0 \quad \text{as} \quad n \to \infty.$$

For arbitrary $p \in \mathbb{N}$ let $q = Mp$. For any $r > q$ take $s \in \mathbb{N}$ with $N^s \geq \frac{r}{q} > N^{s-1}$. Fix a natural number n, $n \geq s + i_0$ and consider first N^s intervals of E_n. Let c_j denote the midpoint of $I_{n,j}$, $j = 1, 2, \ldots, N^s$. Set $f_n(x) = g^q(x)$ where $g(x) = \Pi_{j=1}^{N^s}(x - c_j)$ for $x \in K_N \cap [0, l_{n-s}]$ and $g(x) = 0$ otherwise on K_N. Let us estimate the norms of f_n.

Fix $k \in \mathbb{N}$, $k \leq p$ and $x \in \bigcup_{j=1}^{N^s} I_{n,j}$. By Lemma 1 in [6], we have

$$|f_n^{(k)}(x)| \leq C_{p,r}|g(x)|^{q-k} \tag{1}$$

where $C_{p,r} = \frac{(N^s \cdot q)!}{(N^s \cdot q - k)!} < (N \cdot r)^p$ does not depend on n.

By the structure of K_N we have $|g(x)| < l_n \tau^{N-1}$ where $\tau = l_{n-1} l_{n-2}^N \cdots l_{n-s}^{N^{s-1}}$. Thus

$$|f_n|_p \leq C_{p,r}(l_n \tau^{N-1})^{q-p}. \tag{2}$$

Next we estimate

$$A_p := \frac{|(R_x^p f_n)^{(k)}(y)|}{|x - y|^{p-k}}, \quad k \leq p, x \neq y, x, y \in K_N.$$

If $|x - y| < h_n$, then x, y belong to the same $I_{n,j}$ for some j. Hence applying the Lagrangian form for Taylor's remainder we find $\xi \in I_{n,j}$ such that

$$(R_x^p f_n)^{(k)}(y) = [f^{(p)}(\xi) - f^{(p)}(x)] \frac{(y - x)^{p-k}}{(p - k)!}$$

and so $A_p \leq 2C_{p,r}(l_n \tau^{N-1})^{q-p}$.

If $|x - y| \geq h_n$, then by (1), (2)

$$\begin{aligned} A_p &\leq |f_n^{(k)}(y)| \cdot |x - y|^{k-p} + \sum_{i=k}^{p} |f_n^{(i)}(x)| \cdot \frac{|x - y|^{i-p}}{(i - k)!} \\ &\leq C_{p,r}(l_n \cdot \tau^{N-1})^{q-p} \left[\left(\frac{l_n \cdot \tau^{N-1}}{h_n} \right)^{p-k} + \sum_{i=k}^{p} \frac{1}{(i - k)!} \left(\frac{l_n \cdot \tau^{N-1}}{h_n} \right)^{p-i} \right]. \end{aligned}$$

Since $l_n \cdot \tau^{N-1} < h_n$ we get

$$A_p \leq C_{p,r}(1 + e)(l_n \cdot \tau^{N-1})^{q-p} \quad \text{and} \quad \|f\|_p \leq C_{p,r}(2 + e)(l_n \cdot \tau^{N-1})^{q-p}.$$

Clearly

$$|f_n|_q \geq |f^{(q)}(c_1)| = q!\,|g'(c_1)|^q;$$

$$|g'(c_1)| = \Pi_{j=2}^{N^s}(c_j - c_1) > \left(\frac{l_{n-1}}{N}\right)^{N-1} \cdot \left(\frac{l_{n-2}}{N}\right)^{N^2-N} \cdots \left(\frac{l_{n-s}}{N}\right)^{N^s-N^{s-1}}$$

$$= \tau^{N-1} \cdot N^{1-N^s}.$$

Thus,

$$|f_n|_q \geq q!\,N^{-r\cdot N} \cdot \tau^{(N-1)q}.$$

Analogously, $\|f_n\|_r \leq C_r$.

Finally, we conclude for some D independent of n that

$$\frac{\|f_n\|_p\,\|f_n\|_r^\epsilon}{|f_n|_q^{1+\epsilon}} \leq D l_n^{q-p} \tau^{-(N-1)(\epsilon q+p)}$$

$$= D l_{n-s}^{(\alpha_n\cdots\alpha_{n-s+1})(q-p)} [l_{n-s}^{\alpha_{n-1}\cdots\alpha_{n-s+1}} l_{n-s}^{(\alpha_{n-2}\cdots\alpha_{n-s+1})N} \cdots l_{n-s}^{N^{s-1}}]^{-(N-1)(\epsilon q+p)}.$$

Let us now show that

$$w := (\alpha_n \cdots \alpha_{n-s+1})(q-p)$$
$$- (N-1)(\epsilon q + p)[(\alpha_{n-1}\cdots\alpha_{n-s+1}) + N(\alpha_{n-2}\cdots\alpha_{n-s+1}) + \cdots + N^{s-1}]$$

which is the exponent of l_{n-s}, is positive and bounded away from zero. Indeed for the expression in the square brackets we have

$$[\cdots] \leq \alpha_{n-1}\cdots\alpha_{n-s+1}\cdot\frac{\beta}{\beta-N} \leq \alpha_n\cdots\alpha_{n-s+1}\cdot\frac{1}{\beta-N} =: w_1,$$

therefore

$$w \geq w_1 \cdot [(\beta - N)(q-p) - (N-1)(\epsilon q + p)]$$
$$= w_1 \cdot p \cdot \{M[\beta - N - \epsilon(N-1)] - \beta + 1\} \geq w_1 \cdot p$$

due to the choice of M and ϵ.

So $w \geq \beta^s \frac{p}{\beta - N}$ from which it follows that

$$\lim_{n\to\infty} \frac{\|f_n\|_p\,\|f_n\|_r^\epsilon}{|f_n|_q^{1+\epsilon}} = 0.$$

Therefore the condition (b) is not fulfilled and the compact set K_N does not have the extension property.

The second statement of the theorem can be proved quite similarly to Theorem 3 in [6], so we omit it. ■

Corollary *For a compact set K_N, let the limit $\alpha = \lim \alpha_n$ exist and be not equal to N. Then K_N has the extension property if and only if $\lambda_0(K_N) > 1$.*

Remark The set $K_N^{(\alpha)}$, $\alpha > N$, gives us an example of a perfect set of finite class without extension property (for the definition and details see [18], [6]).

We now turn to the case $N_n \nearrow \infty$. The compact set K_∞ loses its Cantor-type set nature in the following sense: topologically the space $\mathcal{E}(K_\infty)$ is closer to the space $\mathcal{E}(K)$ with K having the form of a sequence of intervals tending to a point than to $\mathcal{E}(K_N)$. Compare the next statement with Theorem 2 in [7].

Theorem 2 *The set K_∞ has the extension property if and only if there exists a constant M such that*

$$l_n \geq h_n^M, \quad \forall n.$$

Proof Assume the space $\mathcal{E}(K_\infty)$ has the property DN. Let us fix p in the condition (b) of Proposition 2. For $\epsilon = 1$ and $q = p + 1$ we find r and C such that (b) is fulfilled.

Defining

$$f_n(x) = \begin{cases} \frac{x^q}{q!} & \text{if } x \in K \cap [0, l_n] \\ 0 & \text{otherwise} \end{cases}$$

we obtain the estimates

$$|f_n|_q \geq 1, \quad \|f_n\|_p \leq 4l_n, \quad \|f_n\|_r \leq 4h_n^{q-r}.$$

Now, Proposition 2 (b) gives $1 \leq 16C \cdot l_n \cdot h_n^{q-r}$, from which the necessity follows.

In order to prove the sufficiency we use a simplified version of Lemma 2 in [7] (see also [8]):

Lemma *Let $K \subset \mathbb{R}$ be a compact set containing $r+1$ distinct points $x_0, x_1, \ldots, x_r$ such that for some h and a constant N*

$$h \leq |x_i - x_j| \leq N \cdot h, \quad i, j = 0, 1, \ldots, r, \quad i \neq j.$$

Then for all $k \leq r$ and $f \in \mathcal{E}(K)$,

$$|f^{(k)}(x_0)| \leq C \cdot h^{-k}|f|_0 + C \cdot h^{r-k}\|f\|_r,$$

where C depends only on r and N.

We will show that

$$\exists\, p\, \forall q\, \exists\, r, C : |\cdot|_q \leq Ct^{Mq} \cdot \|\cdot\|_p + \frac{C}{t^q}\|\cdot\|_r, \quad t > 0, \tag{3}$$

which is equivalent to (a) and hence to the property DN. The constant M here is the same as the one in the statement of the theorem.

Let $p = 0$. Given q, let $r = 2q$. Let n_0 be such that $\frac{N_n}{2} \geq r$ for $n \geq n_0$. Let $t_0 = 1/l_{n_0-1}$. Given $t \geq t_0$, find $n \geq n_0$ such that

$$l_n < \frac{1}{t} \leq l_{n-1}.$$

Let $x_0 \in K$. Then $x_0 \in I_{n,j_0}$ where without loss of generality we may assume that $1 \leq j_0 \leq N_n$. Let $a = a_{n,r+1}$ be the left endpoint of $I_{n,r+1}$. Suppose that $j_0 \leq \frac{N_n}{2}$ and consider the interval $[x_0, x_0 + \frac{1}{2t}]$. (If $j_0 > \frac{N_n}{2}$, then similar arguments apply to $[x_0 - \frac{1}{2t}, x_0]$.)

Case 1: $\frac{1}{2t} \geq a$. Let J be such that the interval $[x_0, x_0 + \frac{1}{2t}]$ intersects $J+1$ basic intervals of the set E_n. Then $J \geq r$ due to the choice of t and $\frac{J}{2}(l_n + h_n) < \frac{1}{2t} < 2J(l_n + h_n)$, as is easy to check. Then $S := [J/r] \geq 1$, the points $x_i = x_0 + iS(l_n + h_n)$, $i = 1, \dots, r$, belong to K and for $i \neq j$, $|x_i - x_j| \geq S(l_n + h_n) > \frac{1}{8rt}$ since $[\frac{J}{r}] > \frac{J}{2r}$ for $J \geq r$. On the other hand, $|x_i - x_j| \leq rS(l_n + h_n) < \frac{1}{t}$, so by the Lemma

$$|f^{(k)}(x_0)| \leq C \cdot t^k |f|_0 + C \cdot t^{k-r} \|f\|_r.$$

Case 2: $\frac{1}{2t} < a$. Now we choose the points x_i, $i = 1, \dots, r$ in the interval I_{n,j_0} in a similar way such that $\frac{l_n}{4r} \leq |x_i - x_j| \leq \frac{l_n}{2}$ for $i \neq j$. Since $a = r(l_n + h_n) \leq 2rh_n \leq 2r \cdot l_n^{\frac{1}{M}}$, by the condition, we can apply the Lemma with $h = \frac{l_n}{4r} > (4r)^{-M-1} \cdot t^{-M}$ and $h < (4rt)^{-1}$:

$$|f^{(k)}(x_0)| \leq C \cdot t^{k \cdot M} |f|_0 + C \cdot t^{k-r} \|f\|_r.$$

We have this estimate for any $x_0 \in K$ and $k \leq q$, thus (3) is proved. ■

We observe that the logarithmic dimension is not related to the extension property of compact sets K_∞ of infinite type as seen by the following proposition and example. But in the finite case we can use it as well for isomorphic classification of corresponding spaces.

Proposition 3 *If K_∞ has the extension property, then $\lambda_0(K_\infty) = \infty$.*

Proof If $\overline{\lim}\alpha_n < \infty$, then clearly $\lim_n \lambda_n = \infty$. So we consider the case of $\overline{\lim}\alpha_n = \infty$. By the previous theorem there is $M \geq 1$ such that $l_n \geq h_n^M$ for all n. Let $I = \{n \in \mathbb{N} : \alpha_n \geq 2M\}$, $J = \mathbb{N} \setminus I$. Then $\lim_{n \in J} \lambda_n = \infty$. For $n \in I$,

$$\begin{aligned} l_n^{1/\alpha_n} &= l_{n-1} \leq 2N_n h_n \leq 2N_n l_n^{1/M} \Rightarrow \frac{1}{2} l_n^{\frac{1}{\alpha_n} - \frac{1}{M}} \leq N_n \Rightarrow \log N_n \\ &\geq \frac{1}{2}\left(\frac{1}{M} - \frac{1}{\alpha_n}\right) \log\left(\frac{1}{l_n}\right) \geq \frac{1}{4M} \alpha_n \cdots \alpha_2 \log\left(\frac{1}{l_1}\right) \Rightarrow \lambda_n \\ &= \frac{\log N_n}{\log \alpha_n} \geq \frac{\log(l_1^{-1})}{4M} \cdot \frac{\alpha_n \cdots \alpha_2}{\log \alpha_n} \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

But the converse of the above proposition is not true as the following example shows:

Example 1 Let $l_n = \exp(-n!)$ and $N_n = [n^{\log n} + 1]$. Then $\alpha_n = n$ and $\lambda_n \geq \log n$. Thus $\lambda_0 = \infty$. But

$$h_n > \frac{l_{n-1}}{e \cdot (N_n - 1)} \geq \frac{1}{n^{\log n} e^{(n-1)!+1}} > \frac{1}{e^{\epsilon \cdot n!}} = l_n^{\epsilon}, \quad n \geq n_0$$

for every $\epsilon > 0$ which means that K_∞ does not have the extension property.

4 Diametral Dimension of $\mathcal{E}(K)$

Approximative and diametral dimensions, introduced by Kolmogorov [9], Pełczyński [14] and Bessaga, Pełczyński and Rolewicz [1], were the first linear topological invariants applicable to isomorphic classification of nonnormed Fréchet spaces. We follow the notation of [11].

Let X be a Fréchet space with a fundamental system of neighborhoods (U_q), let $d_n(U_q, U_p)$ denote the n-th Kolmogorov diameter of U_q with respect to U_p. Then

$$\Gamma(X) = \{(\gamma_n)_{n=0}^{\infty} : \forall p\, \exists q\colon \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}.$$

We will consider the counting function corresponding to the diametral dimension

$$\beta(t) = \beta(U_p, U_q, t) = \min\{\dim L : t \cdot U_q \subset U_p + L\}, \quad t > 0.$$

It is easy to see that $\beta(t) = |\{n : d_n(U_q, U_p) > \frac{1}{t}\}|$, where $|Z|$ denotes the cardinality of the set Z. If X is a Schwartz space and p, q are sufficiently apart, then the function β takes finite values. Clearly, the diametral dimension can be characterized in terms of β in the following way.

Proposition 4 $(\gamma_n) \in \Gamma(X) \Leftrightarrow \forall p\, \exists q : \forall C\, \exists n_0 : \beta(U_p, U_q, C\gamma_n) \le n$ *for* $n \ge n_0$.

But we will directly compare asymptotic behavior of counting functions of isomorphic spaces. The proof of the following proposition is straightforward.

Proposition 5 *If Fréchet spaces X and Y are isomorphic, then*

$$\forall p_1\, \exists p\, \forall q\, \exists q_1, \epsilon > 0 : \beta_Y(V_{p_1}, V_{q_1}, \epsilon t) \le \beta_X(U_p, U_q, t), \quad t > 0.$$

An analogous condition holds after interchanging β_X and β_Y. Here $(V_k)_0^{\infty}$ is a fundamental system of neighborhoods of Y.

Theorem 3 *Let $X = \mathcal{E}(K)$ with $K = K\big((l_n), (N_n)\big)$, let p and q, $p < q$ be fixed natural numbers. If $t \le \frac{1}{5} l_n^{p-q}$, then $\beta(U_p, U_q, t) \le (q+1)N_1 \cdots N_n$. If $t \ge 5(q-p)!\, l_n^{p-q}$, then $\beta(U_p, U_q, t) \ge N_1 \cdots N_n$.*

Proof: Upper bound of β If for some subspace L we have $t \cdot U_q \subset U_p + L$, then $\beta(t) \le \dim L$. Let us fix n such that $5t \le l_n^{p-q}$. Set $M = N_1 \cdots N_n$. In the union $E_n = \bigcup_{k=1}^{M} I_{n,k}$ let $I_{n,k} = [a_k, b_k]$ briefly. For $k = 1, 2, \ldots, M$ and $j = 0, 1, \ldots, q$, let $e_{k,j}(x) = \frac{(x-a_k)^j}{j!}$ if $x \in K \cap I_{n,k}$ and $e_{k,j}(x) = 0$ otherwise on K. We take $L = \mathrm{Span}(e_{k,j})_{k=1, j=0}^{M,q}$. Then $\dim L = (q+1)M$ and it is enough to show that for any function f with $\|f\|_q \le t$ there exists a function $g \in L$ such that $\|f - g\|_p \le 1$. Given $f \in tU_q$ let $g = \sum_{k=1}^{M} \sum_{j=0}^{q} f^{(j)}(a_k) \cdot e_{k,j}$. Clearly, if $x \in I_{n,k}$, then $(f-g)(x) = R_{a_k}^q f(x)$. Since $|x - a_k| \le l_n$ we get

$$|(f-g)^{(i)}(x)| \le \|f\|_q \cdot |x - a_k|^{q-i} \le t \cdot l_n^{q-i}, \quad i \le p. \tag{4}$$

Let now $A_p = \frac{|(R_y^p(f-g))^{(i)}(x)|}{|x-y|^{p-i}}$, $x, y \in K, x \neq y, i \leq p$.

If $x, y \in I_{n,k}$ for some k, then as in [7, p. 569] we can use the following representation

$$R_y^p(f-g)(x) = R_y^q f(x) + \sum_{m=p+1}^{q} (R_{a_k}^q f)^{(m)}(y) \cdot \frac{(x-y)^m}{m!}.$$

Therefore

$$A_p \leq \|f\|_q \cdot |x-y|^{q-i} \cdot |x-y|^{i-p} + \sum_{m=p+1}^{q} \|f\|_q \cdot |y-a_k|^{q-m} \cdot \frac{|x-y|^{m-i}}{(m-i)!} \cdot |x-y|^{i-p}.$$

Here $|x-y| \leq l_n$, hence $A_p \leq t \cdot l_n^{q-p}(1 + \sum \frac{1}{(m-i)!}) < (e+1)t \cdot l_n^{q-p}$.

On the other hand, if the points x, y are situated on different basic intervals of E_n, then $|x-y| \geq h_n \geq l_n$ by assumption. For this case

$$A_p \leq |(f-g)^{(i)}(x)| \cdot |x-y|^{i-p} + \sum_{m=i}^{p} \frac{|(f-g)^{(m)}(y)|}{(m-i)!} |x-y|^{m-i+i-p}.$$

From (4) it follows that $A_p \leq t \cdot l_n^{q-p} + t \cdot l_n^{q-p} \sum_{m=i}^{p} \frac{1}{(m-i)!}$. Thus, $\|f-g\|_p \leq t \cdot l_n^{q-p}(2+e) < 1$, by condition and this establishes the upper bound of β.

Lower bound of β In order to find a lower estimate for Kolmogorov diameters we use the Tikhomirov theorem [20] (see also [11, Proposition 6]): if $d \cdot U_p \cap L \subset U_q$ with $\dim L = n+1$, then $d_n(U_q, U_p) \geq d$.

Therefore, $\beta(t) \geq \dim L$ if $U_p \cap L \subset (1-\epsilon_0)t \cdot U_q$ with some $\epsilon_0 > 0$. Let us take $L = \mathrm{Span}(e_{k,q})_{k=1}^M$ and fix $f = \sum_{k=1}^M C_k \cdot e_{k,q} \in L \cap U_p$. Since $1 \geq \|f\|_p \geq |f^{(p)}(b_k)| \geq |C_k| \frac{l_n^{q-p}}{(q-p)!}$, we have $|C_k| \leq (q-p)!\, l_n^{p-q}$ for all k. Let $x \in I_{n,k}$. Then $|f^{(i)}(x)| \leq |C_k| \frac{l_n^{q-i}}{(q-i)!}$, $i \leq q$, hence $|f|_q \leq (q-p)!\, l_n^{p-q}$.

If $x, y \in I_{n.k}$ then $R_y^q f(x) = 0$. Otherwise, $|x-y| \geq h_n \geq l_n$ and arguing as before, we obtain

$$\begin{aligned} |(R_y^q f)^{(i)}(x)| \cdot |x-y|^{i-q} &\leq |f^{(i)}(x)| \cdot h_n^{i-q} + \sum_{m=i}^{q} \frac{f^{(m)}(y)}{(m-i)!} h_n^{m-q} \\ &\leq \frac{|C_k|}{(q-i)!} + \sum_{m=i}^{q} \frac{|C_{k_1}|}{(m-i)!} \leq (q-p)!\, l_n^{p-q}(1+e). \end{aligned}$$

If we take ϵ_0 with $2+e < 5(1-\epsilon_0)$, then $\|f\|_q \leq (q-p)!\, l_n^{p-q}(2+e) < t \cdot (1-\epsilon_0)$ and $\beta(t) \geq \dim L = M$. ∎

Now we can easily find the diametral dimension of $\mathcal{E}(K)$ for concrete compact set K. In particular for classical Cantor set we have $\beta(U_p, U_q, t) \sim t^{\frac{\log 2}{(q-p)\log 3}}$, that is the diametral dimension of $\mathcal{E}(K)$ is the same as $\Gamma(s)$. Here and subsequently, $F \sim G$ means that for some C, t_0 we have

$$\frac{1}{C}F\left(\frac{t}{C}\right) \le G(t) \le C \cdot F(Ct), \quad t > t_0.$$

The most interesting case we have is for $K = K_N^{(\alpha)}$. Let λ_0 denote as before the logarithmic dimension of K.

Corollary 1 *Let $X = \mathcal{E}(K_N^{(\alpha)})$. Then $\beta(U_p, U_q, t) \sim \log^{\lambda_0} t$, $t \to \infty$.*

Corollary 2 $\Gamma\big(\mathcal{E}(K_N^{(\alpha)})\big) = \{(\gamma_n) : \gamma_n \cdot \exp(-n^{\frac{1}{\lambda_0}}) \to 0$ *as* $n \to \infty\}$.

Corollary 3 *If spaces of the type $\mathcal{E}(K_N^{(\alpha)})$ are isomorphic, then the corresponding compact sets have the same logarithmic dimension.*

Corollary 4 *If $\alpha < N$, then the space $\mathcal{E}(K_N^{(\alpha)})$ is isomorphic to a complemented subspace of s, but is not isomorphic to s.*

Proof Corollary 1 is the result of a simple computation. Corollary 2 follows from Proposition 4, and Corollary 3 does so from Proposition 5.

By definition, $\mathcal{E}(K_N^{(\alpha)})$ is a quotient space of s. If $\alpha < N$, then according to Theorem 1, the space $\mathcal{E}(K_N^{(\alpha)})$ has the property DN. Thus, due to Vogt's characterization (see, *e.g.* [22]) it is a complemented subspace of s. But since $\Gamma\big(\mathcal{E}(K_N^{(\alpha)})\big) \neq \Gamma(s)$, they are not isomorphic.

5 Isomorphic Classification

First we shall generalize Corollary 3 to Theorem 3 for compact sets of finite type.

Let $K_N = K\big((l_n), N\big)$, $K_M = K\big((L_m), M\big)$. Without loss of generality we can take $l_1 = L_1 = 1/e$. Then $l_n = \exp(-\alpha_1 \cdots \alpha_n)$, $L_m = \exp(-\alpha_1' \cdots \alpha_m')$.

Theorem 4 *Let $X = \mathcal{E}(K_N)$, $Y = \mathcal{E}(K_M)$. Suppose that the corresponding limits $\lambda_X = \lim_{n\to\infty} \frac{\log N}{\log \alpha_n}$, $\lambda_Y = \lim_{m\to\infty} \frac{\log M}{\log \alpha_m'}$ exist in the set of extended real numbers and are not equal. Then the spaces X and Y are not isomorphic.*

Proof To be definite, assume that $\lambda_X < \lambda_Y$. If $\lambda_X = 0$, then $\alpha_n \to \infty$ and $\alpha_n' \not\to \infty$. By Theorem 1 in [8], it follows that the space Y has the property $\underline{\text{DN}}$ (for definition see *e.g.* [10]) whereas X does not. So they are not isomorphic.

Consider now the case $0 < \lambda_X < \lambda_Y < \infty$. Suppose, contrary to our claim, that $X \simeq Y$. Then by Proposition 4, for $p_1 = 0$ there exists $p \in \mathbb{N}$ such that for $q = p + 1$ one can find $q_1 \in \mathbb{N}$, $\epsilon > 0$ with

$$\beta_Y(V_0, V_{q_1}, \epsilon t) \le \beta_X(U_p, U_{p+1}, t), \quad t > 0.$$

Assume n and m are large enough and are such that

$$5 \cdot q_1!\, L_m^{-q_1} \leq \frac{\epsilon}{5} l_n^{-1}. \tag{5}$$

Then by taking $t = \frac{1}{5} l_n^{-1}$ and using Theorem 3, we obtain

$$M^m \leq \beta_Y(V_0, V_{q_1}, \epsilon t) \leq \beta_X(U_p, U_{p+1}, t) \leq (p+2)N^n.$$

Set $\alpha = \lim \alpha_n$, $\alpha' = \lim \alpha'_n$. Clearly, $1 < \alpha, \alpha' < \infty$. Take $\rho > 0$ with $\lambda_X = (1-2\rho)\lambda_Y$ and $\delta > 0$ such that

$$\frac{\log(\alpha - \delta)}{\log(\alpha' + \delta)} \cdot \frac{\log \alpha'}{\log \alpha} > 1 - \rho.$$

For this δ one can take n_0, m_0 such that $|\alpha_n - \alpha| < \delta$, $\log \frac{25 q_1!}{\epsilon} \leq \frac{\alpha_1 \cdots \alpha_n}{2}$ for $n \geq n_0$ and $|\alpha'_m - \alpha'| < \delta$, $m \geq m_0$.

Let now

$$C_1 = \frac{1}{2q_1} \frac{\alpha_1 \cdots \alpha_{n_0}}{\alpha'_1 \cdots \alpha'_{m_0}} \frac{(\alpha' + \delta)^{m_0}}{(\alpha - \delta)^{n_0}}, \quad C_2 = \frac{\log C_1}{\log(\alpha' + \delta)},$$

$$m = m(n) = \left[n \cdot \frac{\log(\alpha - \delta)}{\log(\alpha' + \delta)} + C_2 \right].$$

Then

$$(\alpha' + \delta)^m \leq C_1 (\alpha - \delta)^n,$$

and so for $n \geq n_0$, $m \geq m_0$ we get

$$\log \frac{25 \cdot q_1!}{\epsilon} + q_1 \cdot \alpha'_1 \cdots \alpha'_m \leq \alpha_1 \cdots \alpha_n. \tag{6}$$

This is equivalent to (5) and so we have

$$M^m \leq (p+2)N^n \quad \text{or} \quad m \cdot \lambda_Y \cdot \log \alpha' \leq \log(p+2) + n \cdot \lambda_X \cdot \log \alpha.$$

But this contradicts the choice of δ and $m(n)$, as it is easy to check. Thus the spaces X, Y are not isomorphic.

It remains to consider the case $\lambda'_m \to \lambda_Y = \infty$, $0 < \lambda_X < \infty$. Since $(\alpha'_m)^{\lambda'_m} = M$ for all m, we see that there exists m_0 such that $\alpha'_m < 1 + \frac{\log 2M}{\lambda'_m}$ for $m \geq m_0$. Let $\delta < \alpha - 1$ and for $n \geq n_0$ let $\alpha_n \geq \alpha - \delta$. As before the inequality (6) implies the boundedness of $\frac{m}{n}$.

If now $\alpha'_1 \cdots \alpha'_m \nrightarrow \infty$ as $m \to \infty$, then we have (6) for some n_1 and all m, a contradiction. Otherwise let $C_3 = \frac{2q_1 \alpha'_1 \cdots \alpha'_{m_0}}{\alpha_1 \cdots \alpha_{n_0}} \cdot (\alpha - \delta)^{n_0}$. For (6), it is enough to have

$$C_3 \cdot \left(1 + \frac{\log 2M}{\lambda'_m}\right)^{m - m_0} \leq (\alpha - \delta)^n. \tag{7}$$

If now the sequence (m/λ'_m) is bounded, then (7) is valid for an arbitrary m and some fixed n which is a contradiction. If $\overline{\lim}\frac{m}{\lambda'_m} = \infty$, then we take large n, m with $n - 1 < \frac{2\log 2M}{\log(\alpha-\delta)} \cdot \frac{m-m_0}{\lambda'_m} \le n$. Here we have (7) and therefore (6) for some subsequences (m_j), (n_j) but the ratio $\frac{m_j}{n_j}$ together with λ'_{m_j} are unbounded. This contradiction completes the proof. ■

The situation with compact sets of infinite type is much more complicated. By similar arguments we can distinguish now the spaces corresponding to different types. Let $K_\infty = K\big((L_m),(M_m)\big)$, $M_m \nearrow \infty$, $L_1 = 1/e$ and hence as before $L_m = \exp(-\alpha'_1 \cdots \alpha'_m)$. Set $H_m = \frac{L_{m-1} - M_m \cdot L_m}{M_m - 1}$.

Theorem 5 *Let $X = \mathcal{E}(K_N^{(\alpha)})$, $Y = \mathcal{E}(K_\infty)$ and assume the limit $\lambda_Y = \lim_{m\to\infty} \frac{\log M_m}{\log \alpha'_m}$ exists in the set of extended real numbers. Then the spaces X and Y are not isomorphic.*

Proof We may assume that there exists a constant $C_1 \ge 1$, such that $L_m \ge H_m^{C_1}$, $\forall m$. Otherwise by Theorem 3 in [8] the space Y does not have the property $\underline{DN}$ and the result follows. Then we get

$$L_{m-1} < 2M_m \cdot L_m^{\frac{1}{C_1}}$$

and

$$2M_m \ge \exp\left(\alpha'_1 \cdots \alpha'_{m-1}\Big(\frac{\alpha'_m}{C_1} - 1\Big)\right). \tag{8}$$

Suppose that the spaces X, Y are isomorphic. Arguing as above, we see that if m and n satisfy

$$\log \frac{25 \cdot q_1!}{\epsilon} + q_1 \cdot \alpha'_1 \cdots \alpha'_m \le \alpha^n \tag{9}$$

then

$$M_1 \cdots M_m \le (p+2)N^n. \tag{10}$$

Consider the case $\lambda_Y < \infty$. Here $\alpha'_m \to \infty$, with $m \to \infty$ and $M_m < (\alpha'_m)^{2\lambda_Y}$ for large enough m. This contradicts (8).

Let now $\lambda_Y = \infty$. If $\alpha'_m \to 1$, then we take $m = n$ in (9) and obtain a contradiction in (10) as $M_m \nearrow \infty$. It suffices to consider the case $\lambda_Y = \infty$, $\alpha'_m \nrightarrow 1$. For large m let us take $n = C_2 + \frac{\log(\alpha'_1\cdots\alpha'_m)}{\log \alpha}$, where C_2 is fixed such that $n \in \mathbb{N}$ satisfies the condition (9).

From (10) we deduce that

$$(\alpha'_1)^{\lambda'_1} \cdots (\alpha'_m)^{\lambda'_m} \le (p+2)N^{C_2} \cdot (\alpha'_1 \cdots \alpha'_m)^{\frac{\log N}{\log \alpha}},$$

which is impossible as $\lambda'_m \to \lambda_Y = \infty$, $\alpha'_m \nrightarrow 1$. ■

Question Does the space $\mathcal{E}(K_\infty)$ have a basis?

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